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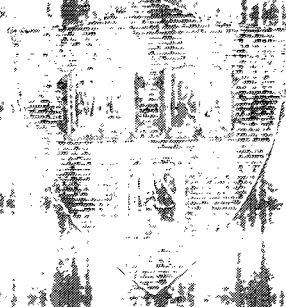
GAP-MINIMAL SYSTEMS OF NOTATIONS
AND THE CONSTRUCTIBLE HIERARCHY[†]

by

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13-72

Center for Research in Computing Technology



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Abstract

If a constructibly countable ordinal α is a gap ordinal, then the order type of the set of index ordinals smaller than α is exactly α . The gap ordinals are the only points of discontinuity of a certain ordinal-valued function.

The notion of gap-minimality for well-ordered systems of notations is defined and the existence of gap-minimal systems of notations of arbitrarily large constructibly countable length is established.

PREFACE

The author would like to express her gratitude to all those who have aided her and have encouraged her research.

Special thanks are due to Professor Ronald V. Book for his constant advice, friendship and support and for his very appropriate editorial comments and to Robert Kirk for his encouragement and willingness to listen to tentative proofs and ideas. George Boolos kindly supplied a copy of his dictorial dissertation. Mrs. Trylla Esherick did a magnificent job of typing the final draft of the paper.

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SYNOPSIS

Using systems of notations we provide a classification of the constructible sets of integers according to their arithmetical degrees.

In the Introduction we present an overview of the main results concerning the hyperarithmetical and the ramified analytical hierarchies and summarize the principal results of this dissertation.

Chapter 0, Preliminaries, presents our choice of notation and contains a few propositions which will be used later on. The proofs of 2.5 and 2.8 are essentially given in [1].

In Chapter I we study a certain class of ordinals, the gap ordinals, and we show that they are evenly distributed among the countable constructible ordinals. That gap ordinals exist was established by Putnam in [13]. They appear for the first time in the literature under this name in [10]. Except the Lemma 1.3, 1.5 and 1.6 which were adapted from [10], all the results in this chapter are new. Theorem 1.10 states that there are exactly α "non-gap" ordinals smaller than any gap ordinal α , and Theorem 2.1 offers a new characterization of the gap ordinals as the only points of discontinuity of a certain ordinal-valued function.

In Chapter II we consider a new minimality requirement for systems of notations and we thus define gap-minimal systems of notations. As a main result we obtain, in Theorem 3.2 that there exists a gap-minimal system of notations containing notations for all ordinals smaller than any arbitrary fixed countable constructible ordinal.

Lemma 1.4 (Recursion Lemma) in this chapter is due to Rogers [16] and is a standard result in higher recursion theory. Lemma 2.1 and Theorems A, B, C (stated without proof) are due to Putnam and Leeds [10]. All the other results in the chapter are new.

The results in Chapter II are in a sense parallel to the results in [10], which provide a classification of constructible sets of integers using generalized admissible degree hierarchies.

INTRODUCTION

In this dissertation we study the arithmetical degrees of sets of integers in the constructible hierarchy. We will show that the hierarchy of constructible sets of integers up to any given level can be considered as an extension of the ramified analytical hierarchy, which in turn can be considered as an extension of the hyperarithmetical hierarchy.

The hyperarithmetical hierarchy provides a classification of a certain collection of sets of integers according to their degrees of unsolvability. Attempts at organizing the sets of integers not covered by this hierarchy have led to classifications of sets of integers according to their arithmetical degrees; the new hierarchy obtained is called the ramified analytical hierarchy. We will show that the ramified analytical hierarchy itself can be extended, using the same methods as those employed in defining the hyperarithmetical hierarchy.

In this introduction we review the known results about these hierarchies and we attempt to justify the results in the paper.

1. The hyperarithmetical hierarchy.

The classical definition of the hyperarithmetical hierarchy is provided by induction on the Kleene-Church system of notations O . O is a set of integers on which a partial ordering $<_O$ is imposed and which is the smallest set of integers satisfying the following conditions:

- (i) $1 \in 0 \ \& \ (x)(x \in 0 \Rightarrow 1 <_0 x \vee x = 1)$
- (ii) $(x)(x \in 0 \Rightarrow 2^x \in 0 \ \& \ (y)(y \in 0 \Rightarrow (y <_0 x \vee y = x \Rightarrow y <_0 2^x)))$
- (iii) $(e)(\phi_e \text{ is an order preserving map} \Rightarrow 3.5^e \in 0 \ \& \ (n)(\phi_e(n_0) <_0 3.5^e)),$

where $0_0 = 1$, $(n+1)_0 = 2^n 0$, and ϕ_e is an order preserving map if and only if

$$(n)(\phi_e(n_0) \text{ convergent} \ \& \ \phi_e(n_0) \in 0 \ \& \ (m)(n)(m < n \Rightarrow \phi_e(m_0) <_0 \phi_e(n_0))).$$

With each element a in 0 is associated a set of integers $H^0(a)$ as follows:

$$H^0(1) = \emptyset$$

$$H^0(2^a) = H^0(a)'$$

$$H^0(3.5^e) = \{J(x, y) \mid y <_0 3.5^e \ \& \ x \in H^0(y)\}.$$

The properties of 0 which are most relevant to this paper are:

(i) 0 satisfies "internal uniqueness," and (ii) 0 is a minimal system of notations. We will write ' $|a|_R$ ' for the ordinal for which the integer a is a notation in R . A system of notations R satisfies internal uniqueness if for all a, b , if $|a|_R = |b|_R$, then $H^R(a) \equiv_T H^R(b)$. A system of notations with this property assigns then a unique Turing degree to all ordinals which have at least one notation in it. The fact that 0 satisfies internal

uniqueness was established by Spector in [17]. A system of notations R is minimal if the Turing degree it associates with an ordinal is as low as possible, i.e. if for any other system of notations S , if $|a|_R = |b|_S$, then $H^R(a)$ is recursive in $H^S(b)$. In [11] it is shown that 0 is a maximal minimal system of notations, in the sense that it contains notations for as large a segment of the classical ordinals as is possible for a minimal system to do. The smallest ordinal which is not assigned a notation in 0 is a countable ordinal, which is known in the literature as "constructive ω_1 " (ω_1^{CK} --Kleene-Church ω_1). In order to define the hyperarithmetical hierarchy we need the concept of "recursive union." If Γ is a class of sets of integers, then the recursive union of Γ is the collection of sets of integers recursive in at least one element of Γ , i.e.

$$RU(\Gamma) = \{A \subseteq \mathbb{N} \mid (\exists B)(B \in \Gamma \ \& \ A \leq_T B)\}$$

Then, the hyperarithmetical hierarchy, H.A., is defined as:

$$H.A. = RU(\{H^0(a) \mid a \in \mathbb{N}\})$$

Kleene has shown in [9] that $H.A. = \Delta_1^1 (= \Sigma_1^1 \cap \Pi_1^1)$.

Two other "hierarchical" definitions of H.A. besides the one given above, are known. One uses degree-hierarchies and the other uses definitions in second-order number theory. It is interesting that H.A. can be extended in three different directions, slightly modifying the three definitions, in such a way that the resulting

class of sets is the same in the three cases. For completeness sake we give here the two other definitions:

(i) Let f be a function from ω_1^{CK} into the family of Turing degrees defined as follows:

1. $f(0) = \deg_T(\emptyset)$
2. $f(\alpha+1) = (\deg_T(f(\alpha)))'$ if $\alpha < \omega_1^{CK}$
3. $f(\lambda) =$ an upper bound to $RU(\{f(\alpha) \mid \alpha < \lambda\})$ and $f(\lambda)$ is recursive in the jump-jump of any other such upper bound, if $\text{Lim}(\lambda)$ and $\lambda < \omega_1^{CK}$.

f is said to be a degree-hierarchy and H.A. can be defined from f as follows:

$$\text{H.A.} = \{A \in N \mid (\exists \alpha)(A \leq_T f(\alpha))\}.$$

(ii) A hierarchy of sets is defined inductively on the ordinals $< \omega_1^{CK}$:

$$A_0 = \{A \in N \mid A \text{ is recursive}\}$$

$$A_{\alpha+1} = \{A \in N \mid A \text{ is definable over } A_\alpha \text{ in second-order number theory using constants from } A_\alpha\}$$

$$A_\lambda = \bigcup_{\alpha < \lambda} A_\alpha \text{ if } \text{Lim}(\lambda)$$

Kleene has shown that:

$$\text{H.A.} = \{A \in N \mid (\exists \alpha)(A \in A_\alpha)\}.$$

2. The ramified analytical hierarchy.

The classical definition of the ramified analytical hierarchy is obtained from the last definition of H.A. above, the one using second-order number theory, by simply dropping the requirement that $\alpha < \omega_1^{\text{CK}}$ in the inductive definition. Then the ramified analytical hierarchy, R.A. is just $\bigcup_{\alpha \in \text{On}} A_\alpha$. It is obvious by a cardinality argument that this new hierarchy of sets must collapse at some ordinal. The point of collapse was proved to be countable by Cohen [4]. This ordinal is called β_0 . Then, $\text{R.A.} = A_{\beta_0}$. It was conjectured by Cohen, and proved independently by Gandy and Putnam (unpublished) that A_{β_0} is the minimum β -model for analysis. The problem was then to try to extend H.A. using its other characterizations (systems of notations and degree-hierarchies) in such a manner that we get exactly R.A. in both cases. Obviously some clause in those definitions had to be weakened. In one case, relaxing the requirement that the system of notations on which the hierarchy is defined to be minimal, has proved most fruitful. In [12], Putnam and Lukas define a system of notations R to be almost-minimal, if for any other system of notations S , if $|a|_R = |b|_S$, then $H^R(a)$ is arithmetical in $H^S(b)$. They prove that there exists an almost-minimal system of notations R which contains notations for all ordinals smaller than β_0 , and that R is maximal almost-minimal, i.e. that no other almost-minimal system of notations can give notations to ordinals bigger than or equal to β_0 . Then they show that

$$R.A. = A_{\beta_0} = RU(\{H^R(a) \mid R \text{ is a maximal almost-minimal system of notations and } a \in R\})$$

In the case of the degree-hierarchy, we modify the third clause in the definition of f , to get an admissible degree-hierarchy as follows:

We will say that S is a uniform upper bound for a countable collection of sets of integers Γ , if Γ is an S -recursively enumerable family of S -recursive sets. Let f be a function from the ordinals into the family of Turing-degrees such that:

1. $f(0) = \deg_T(\emptyset)$
2. $f(\alpha+1) = (\deg_T(f(\alpha)))'$
3. $f(\lambda)$ is a uniform upper bound of $RU(\{f(\alpha) \mid \alpha < \lambda\})$ and there exists an n such that $f(\lambda)$ is recursive in the n th jump of any other such uniform upper bound.

Boyd, Hensel and Putnam show in [3], that such a function f , called an admissible degree hierarchy, exists and can be defined on the initial segment of the ordinals determined by β_0 . Then $R.A.$ can be defined as:

$$R.A. = A_{\beta_0} = \{A \subseteq \mathbb{N} \mid (\exists \alpha)(\alpha < \beta_0 \text{ \& } A \leq_T f(\alpha))\}$$

where f is an admissible degree hierarchy extending up to β_0 .

In his dissertation [1], Bollos shows that if one modifies slightly the definition of the constructible hierarchy, L , (see precise definition in the next chapter) the classification of sets of integers provided by the ramified analytical hierarchy agrees with the classification provided by the constructible hierarchy, i.e. that for all α , if $1 \leq \alpha \leq \beta_0$, then $M_\alpha \cap P(\omega) = A_\alpha$.

The next obvious move is to try getting results about the arithmetical degrees of the constructible sets which occur at levels above β_0 .

3. Gap ordinals and gap-minimal systems of notations.

The fact that three independent hierarchical characterizations of the ramified analytical sets stop at the same ordinal, β_0 , leads one to believe that by studying ordinals similar to β_0 , we can extend the ramified analytical hierarchy even further. This turns out to be the case.

The first part of this thesis is devoted to the study of these ordinals. We take as the most important characteristic of the β_0 -like ordinals the following: No new sets of integers are constructed in the constructible hierarchy at the levels beginning with the successor of the " β_0 -like" ordinals for a certain number of stages (which can be very large). These ordinals, whose exact definition is given in Chapter I, are called gap ordinals. An ordinal α , such

that there exists a set of integers in $M_{\alpha+1} - M_\alpha$, will be called an index ordinal. In Chapter I will show that there are exactly β indices which are smaller than any given β which is a gap ordinal, thus showing that, in a sense, there are enough indices to absorb the gaps, no matter how long the gaps are.

In the second chapter we use gap ordinals to construct a hierarchy of sets extending arbitrarily close to the first constructibly uncountable ordinal, ω_1^L . This is done by relaxing still more the minimality requirement on systems of notations and defining gap-minimal systems of notations. In our main theorem (Chapter II.3.2) we prove that there exists a gap-minimal system of notations up to any ordinal less than constructible ω_1 , and that the classification of sets of integers given by these systems of notations is, in a sense to be made precise, best possible.

Assuming the Axiom of Constructibility ($V = L$) to be true, we can claim that we have a classification of all sets of integers according to their arithmetical degree. Of course, there is no reason (as of now) to believe this axiom to be true, so all we have is a classification of the constructible sets of integers.

Two questions which are not touched upon in the thesis but should be considered are the following:

A. The connection between gap ordinals and admissible ordinals (in the sense of metarecursion theory).

B. The gap-minimal systems of notations constructed in Chapter II are not the most general systems of notations, since they are well-orderings, rather than well-founded partial-orderings with a unique least element. The problem is then, are there any (tree-like) gap-minimal systems of notations which extend arbitrarily far up to ω_1^L ? What additional properties would such systems of notations have to have?

These questions will be investigated in another paper.

As a final point of the Introduction we would like to point out that this thesis is but one link in a chain of papers written by Professor Hilary Putnam and his students, and that the results in it are best understood if considered in conjunction with the papers [2], [3], [10], [11], [1], listed in the bibliography.

Chapter 0.

PRELIMINARIES

In this chapter we discuss our choice of notation and present some definitions and results which will be used in the following chapters.

1. Notation.

We will use the standard notations in recursive function theory, as they appear in Rogers [16]. The e -th A -partial recursive function will be denoted by ϕ_e^A , the domain of ϕ_e^A , i.e. the set $\{x \mid \phi_e^A(x) \text{ convergent}\}$ will be denoted by W_e^A . J will be a fixed primitive recursive 1-1 function from $N \times N$ onto N ; its inverse functions will be denoted by K and L such that

$$KJ(x,y) = x \quad \text{and} \quad LJ(x,y) = y$$

Jumps are defined as follows: let S be an arbitrary set of integers. Then

$$S^{(0)} = S$$

$$S^{(n+1)} = \{x \mid \phi_x^{S^{(n)}}(x) \text{ convergent}\} \quad (= (S^{(n)})')$$

$$S^{(\omega)} = \{J(x,y) \mid x \in S^{(y)}\}$$

Notation in set theory is fairly standard and we will not diverge from it. All our set theoretic constructions can be formalized in Zermelo-Fraenkel (ZF) set theory plus the Axiom of Choice. (See Mendelson [14], for the axioms of ZF.)

ON is the class of ordinal numbers and each $\alpha \in \text{ON}$ is identified with the set of all smaller ordinals, $\{\beta: \beta < \alpha\}$. Hence the ordinal 0 is \emptyset , $\alpha+1$ is $\alpha \cup \{\alpha\}$, and λ is a limit if and only if $\lambda \neq 0$ and $\bigcup \lambda = \lambda$. We use $\alpha, \beta, \gamma, \dots$ to range over ON.

Sometimes we look at results in second order number theory; in that case, small Roman letters will represent integers, capital Roman letters will represent sets of integers. When we are proving results in set theory, small Roman letters will represent arbitrary sets. Although these notations are not consistent with each other, we use them, since they appear as such throughout the literature. However, it will always be clear from context which case we are in at a particular moment.

2. The constructible hierarchy.

In [8], Gödel defines a transfinite hierarchy M_α of sets and uses it to prove the consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis with the axioms of ZF. Since we will examine classes of this hierarchy in detail, we define it here.

2.1 Definition. For all $\alpha \in \text{ON}$, let M_α be as follows:

$$M_0 = \{\emptyset\}$$

$$M_{\alpha+1} = \{X \mid \text{Fodo}(X, M_\alpha)\}$$

$$M_\lambda = \bigcup_{\alpha < \lambda} M_\alpha$$

where 'Fodo(A,B)' means that there is a formula $\phi(X, X_0, \dots, X_n)$ of set theory and elements b_0, \dots, b_n of B such that

$$A = \{X \in B \mid \phi^B(X, b_0, \dots, b_n)\},$$

where ϕ^B is ϕ , with all the quantifiers restricted to B.

The union of all the classes of the constructible hierarchy,

$\bigcup_{\alpha \in ON} M_\alpha$ is denoted in the literature by L and is referred to as the constructible universe.

2.2 Definition. A set x is said to be constructible, if $x \in L$, in other words if there exists an α , such that $x \in M_\alpha$.

The following properties of the M_α 's will be of use later on. Their proofs are standard, and may be found for example in Felgner [7].

2.3 Proposition. For all α ,

(i) M_α is transitive, i.e. every element of an element of M_α is an element of M_α itself.

(ii) $\alpha \in M_{\alpha+1}$

For all α, β ,

(iii) If $\alpha < \beta$, then $M_\alpha \subsetneq M_\beta$.

2.4 Definition. A set A of integers is of order α , if $A \in M_{\alpha+1} - M_\alpha$. An ordinal α , with the property that there exists at least one set of integers of order α , will be called an index.

We will abbreviate ' α is an index' by $I(\alpha)$. It is part of the content of Gödel's proof mentioned above that there are no indices greater than or equal to ω_1^L (ω_1^L is the first uncountable constructible ordinal). Putnam has shown in [15], that there are L -uncountably many non-indices. Clearly there are L -uncountably many indices. If K is a set of order α , we will call K (arithmetically) complete (of order α) if every set of integers in $M_{\alpha+1}$ is arithmetical in K , i.e.

$$(A)(A \in M_{\alpha+1} \cap P(\omega) \Rightarrow (En)(A \leq_T K^{(n)})$$

It is not hard to see that if $I(\alpha)$, then $I(\alpha+1)$. We include the proof here for two reasons: (i) We will make extremely frequent uncited use of this fact throughout the paper, and (ii) we offer a concrete example of how one defines a set of $M_{\alpha+1}$ over M_α .

2.5 Proposition. If $A \in M_\alpha \cap P(\omega)$, then $A^{(\omega)} \in M_{\alpha+1} \cap P(\omega)$, for $\alpha > 0$.

Proof. We follow the proof in Boolos [1].

Let $Q(A, i) = \{J(x, i) \mid x \in A\}$. Observe that for any $\alpha > 0$, if A and B are in M_α , so are $A \cup B$, $Q(A, i)$ for any integer i , and

$B' = \{x \mid (Ey)T_1^B(x, x, y)\}$. Hence for any A in M_α , $\bigcup_{j=0}^i Q(A^{(j)}, j)$ is in M_α , for any integer i . Then,

$$\begin{aligned} A^{(\omega)} &= \{x \in M \mid (ER)((y)(J(y, 0) \in R \iff y \in A) \ \& \\ &\quad (i)(0 \leq i \leq Lx \implies (J(y, i+1) \in R \iff (Ez)T_1^{\{w \mid J(w, i) \in R\}}(y, y, z) \ \& \\ &\quad \& \ x \in R)))\} = \\ &= \{x \in M \mid (ER)((y)(J(y, 0) \in R \iff y \in A) \ \& \\ &\quad (i)(0 \leq i \leq Lx \implies J(y, i+1) \in R \iff \\ &\quad (Ez)(T_1^1(z, y, y) \ \& \ (w)_{<1h(z)} ((J(w, i) \in R \implies (z)_w = 1 \ \& \\ &\quad (J(w, i) \notin R \implies (z)_w = w)))))) \ \& \ x \in R\}. \end{aligned}$$

The T 's used in the proof refer to the familiar Kleene T -predicates. Although the expression for $A^{(\omega)}$ above looks somewhat formidable, the idea behind it is rather simple. An integer will be in the desired set, just in case it belongs to some finite jump of A . Moreover, we can get to that finite jump by taking the single jump as often as necessary.

Generally, it is easier to speak about functions of integers than to speak about relations between arbitrary sets. This led to the notion of an arithmetical copy of a collection of sets, copy in the sense that the relations existent among members of the given collection of sets are preserved under the function which does the copying. We define precisely the notion of arithmetical copy just for

the M_α 's, since this is the only case in which we use it. The general definition can be found in [12].

Let F be a set of integers. Then

$$\text{FIELD}(F) = \{x \mid (\exists y)(J(x,y) \in F \vee J(y,x) \in F)\}.$$

2.6 Definition. Let F be a set of integers and let $\alpha \in \text{ON}$. F is said to be an arithmetical copy of M_α , if there exists a function f , not necessarily in L , such that f is one-to-one from $\text{FIELD}(F)$ onto M_α , and such that

$$(x)(y)(J(x,y) \in F \iff f(x) \in f(y)).$$

If for some integer a , every element of $\text{FIELD}(F)$ is of the form $J(x,a)$, i.e. if

$$(x)(x \in F \iff LKx = LLx = a),$$

then F is said to be an a-initial arithmetic copy of M_α . The last definition, of an initial copy, is extremely useful, since it enables us to distinguish between copies of various M_α 's.

The fundamental result in Boolos [1], is that if $I(\alpha)$, then there exists an arithmetical copy of M_α , which is of order α . It follows that for any a , there exists an a-initial arithmetical copy of M_α of order α .

2.7 Proposition. If E_α is an arithmetical copy of M_α of order α , then E_α is a complete set of order α .

The proof of the above proposition although straightforward is quite tedious and we omit it here. The idea behind it is as follows: Any set of integers in $M_{\alpha+1}$ is first-order-definable over M_α ; any such definition can be 'translated' into a definition over E_α using only number quantifiers.

Note that if A is any complete set of order α , so are all the finite jumps of A , thus there is no unique Turing degree characterizing the complete sets of a given order. This is the main reason for which we concentrate our attention to arithmetical degrees and in Chapter II we even modify the definition of a jump to mean "arithmetical jump."

2.8 Proposition. Let E_α be an initial arithmetical copy of M_α of order α . Then there exists an arithmetical copy of $M_{\alpha+1}$ of order $\alpha+1$.

The proof of Proposition 2.8 forms Chapter II of [1]. Again we indicate the main steps: If ϕ is a formula with number quantifiers, containing $=, \epsilon$, and constants from $\text{Field}(E_\alpha)$, we write ' $E_\alpha \models \phi$ ' for what is expressed in model theory as:

$$\langle \text{Field}(E_\alpha), \{ \langle x, y \rangle \mid J(x, y) \in E_\alpha \}, \text{Field } E_\alpha \rangle \models \phi,$$

where satisfaction (\models) is defined by:

$$(i) E_\alpha \models a \in b \iff J(a, b) \in E_\alpha$$

$$(ii) E_\alpha \models (x)\phi x \iff (a \in \text{Field}(E_\alpha))(E_\alpha \models \phi[a/x])$$

etc.

If ϕ contains the constants a_1, \dots, a_n , and a_i represents the set x_i in M_α , then

$$E_\alpha \models \phi \iff M_\alpha = \psi,$$

where ψ is like ϕ , except that it has x_i 's where ϕ had a_i 's. Then every formula of this kind containing one free variable x corresponds to a first-order definition over M_α of a member of $M_{\alpha+1}$. But the set defined by ϕ will be new, i.e. will be an element of $M_{\alpha+1} - M_\alpha$, just in case

$$E_\alpha \models \sim(Ez)(w)(\phi w \iff w \in z)$$

If y is a Gödel number of such a formula ϕ , y is a least Gödel number of ϕ , just in case

$$(z)(z < y \ \& \ z \text{ is a Gödel number of } \phi \Rightarrow E_\alpha \models \sim(w)(\phi w \Rightarrow \psi w))$$

Suppose E_α is an a -initial arithmetical copy of M_α . Pick an integer $b \neq a$. Then if y is a least Gödel number for ϕ , we let $J(y, b)$ be a name for the set defined by ϕ , i.e. add to E_α , $\{J(d, J(y, b)) \mid E_\alpha \models \phi d\}$. Then the set

$E_\alpha \cup \{J(d, J(y, b)) \mid y \text{ is the least Gödel number of some formula } \phi \text{ and } \phi \text{ defines a new set and } d \in \text{Field}(E_\alpha) \text{ and } E_\alpha \models \phi d\}$

is an arithmetical copy of $M_{\alpha+1}$. To show that the arithmetical copy built above is of order $\alpha+1$ it is enough to observe that the whole procedure is uniformly recursive in $E_\alpha^{(\omega)}$ and b . This copy will be designated by $(E_\alpha, b)^*$, to show the dependence mentioned above. The operation $*$ will be used in the construction of complete sets.

3. Uniform upper bounds.

Let C be a countable collection of sets of integers. A set of integers A is said to be a uniform upper bound of C just in case there exists an A -recursive function f , such that for all sets of integers B ,

$$B \in C \iff (\exists n)(\chi_B = \phi_{f(n)}^A) \& \cdot (n)(\exists B \in C)(\phi_{f(n)}^A = \chi_B)$$

where χ_B is the characteristic function of the set B . Observe that the uniform upper bound of a collection of sets C cannot be itself in C , if C is closed under the jump operation.

3.1 Proposition. Let A be a uniform upper bound for C and let $A \leq_a B$. Then some finite jump of B is also a uniform upper bound of C .

Proof. Observe first that $A \leq_a B \iff (En)(A \leq_1 B^{(n)})$. We need the following fact for the proof of the proposition: If $A \leq_1 B$ via recursive function ϕ_z and if f is an A -recursive function with A -index e_0 , then there exists a recursive function h , such that $h(e_0, z)$ is a B -index for f . This fact is proved as follows: For any x , start computing $f(x)$ using the algorithm for it given by $\phi_{e_0}^A$, and replace every question of the form "Is y in A ?" by the question "Is $\phi_z(x)$ in B ?". The value computed will be unchanged but we have used an oracle for B instead of one for A . The procedure is clearly uniform in z and e_0 . Therefore there exists a recursive h such that

$$f = \phi_{e_0}^A = \phi_{h(e_0, z)}^B.$$

Now let $D \in C$ be arbitrary and suppose f is the A -recursive function generating A -indices of the sets in C . Then for some n , $\chi_D = \phi_{f(n)}^A$. But we know that there exists an integer k such that $A \leq_1 B^{(k)}$. Then $\chi_D = \phi_{h(f(n), z)}^{B^{(k)}}$, where h is the function described in the proof above and z is an index of the recursive function g , s.t. $x \in A \iff g(x) \in B^{(k)}$, where g is 1-1. h is a $B^{(k)}$ -recursive function. Conversely, consider $\phi_{h(f(n), z)}^{B^{(k)}}$ for arbitrary n . By the fact proved above, it equals $\phi_{f(n)}^A$ and is therefore the characteristic function of some element of C . Therefore $B^{(k)}$ is a uniform upper bound for C and the function generating $B^{(k)}$ -indices for C is $\lambda n h(f(n), z)$.

4. Systems of notations.

We assume known all the information about the system of notations O , for example at the level in Rogers' Chapters XI and XVI [16]. In particular we assume known Spector's result about H-sets corresponding to notations for the same ordinal being Turing-equivalent and all the known uniformity results. Since the systems of notations which will be needed in subsequent proofs are more general (or, from a different point of view, weaker) than O , we will study them at the time they are introduced.

4.1 Definition. Let $\alpha \in ON$. Then α is said to be HYP, written $HYP(\alpha)$, just in case

$$(A)(A \in M_\alpha \cap P(\omega) \Rightarrow O^A \in M_\alpha \cap P(\omega)).$$

5. The Ramified Analytical Hierarchy.

In this section we define yet another hierarchy and we state some of its properties. The ramified analytical hierarchy was defined for the first time by Kleene in [9].

5.1 Definition. For all $\alpha \in ON$, let A_α be as follows:

$$A_0 = \{S \mid S \subseteq \mathbb{N} \text{ and } S \text{ is arithmetical}\}$$

$$A_{\alpha+1} = \{S \mid S \subseteq \mathbb{N} \text{ and } S \text{ is definable in second-order number theory using set constants representing sets in } A_\alpha\}$$

$$A_\lambda = \bigcup_{\alpha < \lambda} A_\alpha.$$

While it is still the case that for all α, β , if $\alpha < \beta$, then $A_\alpha \subseteq A_\beta$, this inclusion is not always strict, i.e. there exists an ordinal α such that $A_\alpha = A_{\alpha+1}$. The first ordinal at which this happens is denoted in the literature by β_0 . Cohen has shown in [4] that β_0 is countable. A_{β_0} is a β -model for analysis (that is, it satisfies all the second-order comprehension schemata). Cohen conjectured that A_{β_0} is the minimum β -model for analysis. His conjecture was shown to be true by Putnam and Gandy. (An ω -model \mathcal{M} is a β -model just in case the predicate "For all α , α is not an infinitely descending path in R " holds of a linear ordering R in \mathcal{M} , only if R is actually a well-ordering.)

Our interest in this hierarchy comes from two directions: One is that it is known that β_0 is also the smallest ordinal which is not an index, and the other is Boolos' result that if one alters slightly the definition of the constructible hierarchy, (namely replacing M_0 by the class of the hereditarily finite sets), the two hierarchies agree in their classifications of sets of integers up to and including β_0 , i.e. for all $1 \leq \alpha \leq \beta_0$, $M_\alpha \cap P(\omega) = A_\alpha \cap P(\omega)$.

Chapter I

GAP ORDINALS

In this chapter we will study those ordinals which are not indices. We will show that these ordinals can be characterized in a new way as being the points at which a certain function is discontinuous. As a corollary we will show that indices are 'evenly' distributed among the ordinals which are smaller than ω_1^L .

Section 1

1.1 Definition. Let $\alpha \in ON$. α will be said to be a gap ordinal (abbreviated as $G(\alpha)$) if:

- (i) α is a limit of indices
- (ii) $\neg I(\alpha)$.

1.2 Definition. Let $G(\alpha)$. The ordinal β such that for all $\gamma < \beta$, $M_{\alpha+\gamma+1} - M_{\alpha+\gamma}$ contains no set of integers and $I(\alpha+\beta)$ is called the length of the gap at α . We will denote this uniquely determined ordinal by $g(\alpha)$.

It follows directly from the definitions above that any limit ordinal which is not a limit of indices lies between α and $\alpha+g(\alpha)$ for some α such that $G(\alpha)$. On the other side, no restriction is made on $\alpha+g(\alpha)$, except that it must be an index. By a cardinality

argument we see that if $G(\alpha)$, then there exists a β , such that $I(\beta)$ and $\alpha < \beta$; conversely, if $I(\alpha)$ then there exists a β , such that $G(\beta)$ and $\alpha < \beta$.

We prove now a few properties of gap ordinals which will be used in subsequent proofs.

1.3 Proposition. For all α , if $G(\alpha)$ then $HYP(\alpha)$.

Proof. Observe first that if $G(\alpha)$ then $M_\alpha \cap P(\omega)$ is closed under many operations. Some of these closure properties come from the simple fact that any gap ordinal is a limit ordinal. In particular $M_\alpha \cap P(\omega)$ is closed under definition in second-order number theory; for any definition of a set of integers in second-order number theory can be easily translated to a definition of the same set over some M_β , $\beta < \alpha$. $M_\alpha \cap P(\omega)$ is therefore an ω -model for analysis. Suppose now that $G(\alpha)$ and $A \in M_\alpha \cap P(\omega)$. Then $\alpha \geq \omega_1^A$, for otherwise the relativized hyperarithmetic hierarchy would produce new sets of integers at $M_{\alpha+1}$. In fact $\alpha > \omega_1^A$, for if $\alpha = \omega_1^A$, then 0^A would be of order α , and α would be an index. It follows that $HYP(\alpha)$.

As a corollary to the above proof we see that $M_\alpha \cap P(\omega)$ is in fact a β -model for analysis.

We mentioned before that the first non-index is denoted in the literature by β_0 . Therefore $G(\beta_0)$ and for all α , $\alpha < \beta_0 \Rightarrow I(\alpha)$. It is known that the length of the gap at β_0 , $g(\beta_0)$ is 1. There-

fore $I(\beta_0+1), \dots, I(\beta_0+n), \dots, I(\beta_0+\omega), \dots$, and the β_0 th index is just β_0+1 , hence there are exactly β_0 indices smaller than β_0 . We will show that this is in fact true for all gap ordinals, i.e. that for any $G(\alpha)$, there are exactly α indices smaller than α . In order to do this we define a function O as follows:

1.4 Definition. Let $O: ON \rightarrow ON$ be the ordinal-valued function which associates with every ordinal α , the order of the α th complete set.

Observe that if $O(\alpha) = \beta$, then β is the α th index. Also, for all α , if $\alpha \in \text{Range}(O)$, then $I(\alpha)$. From the definition it follows that:

- (i) O is a monotone function
- (ii) O is 1-1
- (iii) for all α , $O(\alpha) \geq \alpha$.

Clearly O is the identity function on all ordinals smaller than β_0 , and at β_0 , O skips one level, i.e. $O(\beta_0) = \beta_0+1$. Since $I(\alpha) \Rightarrow I(\alpha+1)$, we have $O(\beta_0+n) = \beta_0+n+1$ for all $n < \omega$, and $O(\beta_0+\omega) = \beta_0+\omega$. Hence O has a fixed point at $\beta_0+\omega$, and it continues to be the identity for all the ordinals smaller than the next gap ordinal. These considerations led us to the following conjecture: If $G(\alpha)$, then $\alpha+\omega \cdot g(\alpha)$ is a fixed point of O . The conjecture turns out to be true. In order to prove it we need a whole sequence of lemmas, some of which are interesting in their own right, and some of which are not.

1.5 Lemma. If α is a limit of indices and M_α contains a well-ordering of integers of length α , then $I(\alpha)$. The proof of this lemma is easy and can be found in detail in Putnam & Leeds [10].

It follows from this lemma that if $G(\alpha)$, then M_α cannot contain a well-ordering of integers of length α .

1.6 Lemma. Let $G(\alpha)$. For any β , if $\beta < \alpha$, then $\beta + \beta < \alpha$.

Proof. By Lemma 1.5 it follows that there is no well-ordering of integers of length α in M_β , for arbitrary $\beta < \alpha$. Suppose $I(\beta)$, $\beta < \alpha$ and $\beta + \beta \geq \alpha$. Let E_β be an arithmetical copy of M_β of order β . Then $R = \{J(x, y) \in E_\beta \mid x \text{ and } y \text{ are 'ordinals'}\}$ is a well-ordering of length β , and

$$R' = \{J(x, y) \mid Lx \in \text{Field}(R) \ \& \ Ly \in \text{Field}(R) \ \& \ .Kx = 1 \vee \\ Kx = 2 \ \& \ .Ky = 1 \ \& \ Ky = 2 \ \& \ .Kx = Ky \Rightarrow \\ Kx <_R Ly\}$$

is a well-ordering of integers of length greater than α , and R' is contained in M_δ for some index $\delta < \alpha$. But this implies that there is a well-ordering of integers of length α in M_α , contradicting the fact that $G(\alpha)$. Therefore such a β does not exist.

1.7 Theorem. Let α_1 and α_2 be such that $G(\alpha_1)$ and $G(\alpha_2)$ and such that for all β , $\alpha_1 < \beta < \alpha_2$, either $I(\beta)$ or $\beta < \alpha_1 + g(\alpha_1)$ (i.e. α_1 and α_2 are two successive gap ordinals). Then if $O(\alpha_1) = \alpha_1 + g(\alpha_1)$, the function O has a fixed point δ , $\alpha_1 + g(\alpha_1) < \delta < \alpha_2$.

Proof. The proof will be constructive; we will exhibit one such fixed point. Claim that $\delta = \alpha_1 + \omega \cdot g(\alpha_1)$ is a fixed point of the function O . Using Lemma 1.6 we observe that for all n , $\alpha_1 + n \cdot g(\alpha_1) < \alpha_2$. So $\alpha_1 + \omega \cdot g(\alpha_1)$ is at worst equal to α_2 . But we will show that δ is in the range of O , therefore it cannot be a gap ordinal. So δ belongs to the desired interval. Now observe that

$$\begin{aligned} O(\alpha_1) &= \alpha_1 + g(\alpha_1) \\ O(\alpha_1 + 1) &= \alpha_1 + g(\alpha_1) + 1 \\ &\vdots \\ O(\alpha_1 + \omega) &= \alpha_1 + g(\alpha_1) + \omega; \end{aligned}$$

We will show by induction on $\beta \leq g(\alpha_1)$ that

$$(*) \quad O(\alpha_1 + \omega \cdot \beta) = \alpha_1 + g(\alpha_1) + \omega \cdot \beta.$$

The case $\beta = 0$ is one of the hypotheses of the theorem. Suppose

(*) holds for all $\gamma < \beta$. Then if for some γ , $\beta = \gamma + 1$, we have:

$$0(\alpha_1 + \omega \cdot \gamma) = \alpha_1 + g(\alpha_1) + \omega \cdot \gamma, \text{ by Induction Hypothesis}$$

$$\vdots$$

$$\begin{aligned} 0(\alpha_1 + \omega \cdot \gamma + \omega) &= \alpha_1 + g(\alpha_1) + \omega \cdot \gamma + \omega = \\ &= \alpha_1 + g(\alpha_1) + \omega(\gamma + 1) = \\ &= \alpha_1 + g(\alpha_1) + \omega \cdot \beta. \end{aligned}$$

But the leftmost expression in the last chain of equalities is just $0(\alpha_1 + \omega \cdot \beta)$, so we are done.

Suppose now that $\text{Lim}(\beta)$. Then $0(\text{lub}\{\alpha_1 + \omega \cdot \gamma \mid \gamma < \beta\}) = 0(\alpha_1 + \omega \cdot \beta)$. Let $\sigma = \text{lub}\{0(\alpha_1 + \omega \cdot \gamma) \mid \gamma < \beta\}$. We have to show that $\sigma = 0(\alpha_1 + \omega \cdot \beta)$. σ is clearly $\leq 0(\alpha_1 + \omega \cdot \beta)$ and by the induction hypothesis $\sigma = \alpha_1 + g(\alpha_1) + \omega \cdot \beta$. Suppose $\sigma < 0(\alpha_1 + \omega \cdot \beta)$. Since $\alpha < \alpha_2$, $I(\sigma)$, so σ is in the range of 0 . Hence there exists τ , such that $\sigma = 0(\tau)$ and there exists γ_0 such that $\tau < \alpha_1 + \omega \cdot \gamma_0$, since $\alpha_1 + \omega \cdot \beta$ is a limit ordinal. Then $0(\tau) < 0(\alpha_1 + \omega \cdot \gamma_0) \leq \sigma$. The first inequality holds by monotonicity of 0 and the second one by the definition of σ . So $\sigma < \sigma$, which is clearly a contradiction. Therefore we have:

$$\begin{aligned} 0(\alpha_1 + \omega \cdot g(\alpha_1)) &= \alpha_1 + g(\alpha_1) + \omega \cdot g(\alpha_1) = \\ &= \alpha_1 + (1 + \omega) \cdot g(\alpha_1) = \\ &= \alpha_1 + \omega \cdot g(\alpha_1). \end{aligned}$$

Hence δ is a fixed point of 0 and the theorem is proved.

1.8 Corollary. If α_1 and α_2 are as in the hypothesis of Theorem 1.7, then $O(\alpha_2) = \alpha_2 + g(\alpha_2)$.

Proof. It is easy to see that once the function O hits a fixed point below a gap ordinal, each point above the fixed point but below the next gap ordinal is itself a fixed point of O . Suppose now that $O(\alpha_2) > \alpha_2 + g(\alpha_2)$. But $\alpha_2 + g(\alpha_2)$ is by definition an index, so there must be some ordinal smaller than α_2 , call it γ , such that $O(\gamma) = \alpha_2 + g(\alpha_2)$. But by Theorem 1.7 $O(\gamma)$ can be at most equal to α_2 . So we have $\alpha_2 + g(\alpha_2) \leq \alpha_2$. Since $g(\alpha_2) > 0$, this is clearly a contradiction.

1.9 Corollary. If $G(\alpha)$, then $O(\alpha) = \alpha + g(\alpha)$.

Proof. By induction on the sequence of gap ordinals, let α be a gap ordinal. Suppose that for all β , if $\beta < \alpha$ and $G(\beta)$ then $O(\beta) = \beta + g(\beta)$.

Case (i). $\alpha = \beta_0$. Then $O(\beta_0) = \beta_0 + 1 = \beta_0 + g(\beta_0)$.

Case (ii). There exists a greatest gap ordinal β , which is smaller than α . By the induction hypothesis, $O(\beta) = \beta + g(\beta)$. Then by Corollary 1.8, $O(\alpha) = \alpha + g(\alpha)$.

Case (iii). $\alpha \neq \beta_0$ and there is no greatest gap ordinal smaller than α , i.e. α is a limit of gap ordinals. Clearly $O(\alpha) \geq \alpha + g(\alpha)$. Suppose $O(\alpha) \neq \alpha + g(\alpha)$. Then there exists an ordinal $\beta < \alpha$, such

that $0(\beta) = \alpha + g(\beta)$, and $G(\beta)$. But by the induction hypothesis $0(\beta) = \beta + g(\beta)$ and since $\beta < \alpha$, $\beta + g(\beta) < \alpha$, so $0(\alpha) < \alpha$, which is a contradiction.

Therefore we have proved:

1.10 Theorem. There are exactly α indices less than any gap ordinal α .

Section 2

In this section we study the continuity of the function 0 and we offer a different characterization of gap ordinals.

2.1 Theorem. Let $\alpha \in ON$ and let $\text{Lim}(\alpha)$. Then 0 is continuous at α if and only if $\neg G(\alpha)$. In other words, the gap ordinals are the only points of discontinuity of the function 0 .

By "continuous" we mean "continuous with respect to the natural topology induced by the closure under the least-upper-bound operation."

Proof. In order to prove continuity of a function from the ordinals to the ordinals we have to show that the function commutes with the least-upper-bound-operation at limit ordinals.

We show first that 0 is discontinuous at gap ordinals. Let $G(\alpha)$.

Claim. $\text{lub}\{0(\delta) \mid \delta < \alpha\} = \alpha$.

Proof. If $\delta < \alpha$, $0(\delta) < 0(\alpha)$, by monotonicity of 0 . Therefore the lub of the set considered is less than or equal to $0(\alpha)$. But by Corollary 1.9, $0(\alpha) = \alpha + g(\alpha)$ and every $0(\delta)$ for $\delta < \alpha$, being an index ordinal, is strictly less than α . So the desired lub has to be at most equal to α . But if it is not exactly α , since $\text{Lim}(\alpha)$, there exists $\rho < \alpha$, such that $0(\alpha) = \rho$, contradicting Corollary 1.9, the monotonicity of 0 and the fact that for all α , $0(\alpha) \geq \alpha$. So the claim is true. But $0(\text{lub}\{\delta \mid \delta < \alpha\}) = 0(\alpha) \neq \alpha = \text{lub}\{0(\delta) \mid \delta < \alpha\}$. Therefore 0 is not continuous at α . By taking contrapositives the only if part of the theorem follows.

To prove the other direction of the theorem, let α be such that $\text{Lim}(\alpha)$ and $\neg G(\alpha)$. Then there are two cases to be considered:

- (i) $(EB)(G(\beta) \ \& \ \beta < \alpha < \beta + g(\beta))$, i.e. lies inside the gap of some gap ordinal.
- (ii) $I(\alpha)$.

Case (i). $0(\text{lub}\{\delta \mid \delta > \alpha\}) = 0(\alpha)$. Let $\tau = \text{lub}\{0(\delta) \mid \delta < \alpha\}$.

Clearly $\tau > \beta + g(\beta)$, where β is the ordinal inside whose gap α lies, and $\tau \leq 0(\alpha)$. Suppose $\tau < 0(\alpha)$. Then claim that $I(\tau)$. For if not, either $G(\tau)$, or τ itself lies inside a gap. i.e.

$(\exists \beta)(G(\gamma) \ \& \ \gamma < \tau < \gamma + g(\gamma))$. In the first subcase $\text{lub}\{0(\delta) \mid \delta < \tau\} = \tau$, by the claim proved at the beginning of the proof of the theorem, and since $\tau > \beta + g(\beta)$, $\alpha < \beta + g(\beta) \Rightarrow \alpha < \tau$, so $0(\alpha) < \tau$, which is a contradiction. In the second subcase, $\text{lub}\{0(\delta) \mid \delta < \alpha\} = \gamma$, and $\alpha < \gamma$, so again $0(\alpha) < \gamma < \tau$, the same contradiction. So if our assumption is correct, we have $I(\tau)$. So there exists $\sigma < \alpha$, such that $0(\sigma) = \tau$. But $\text{Lim}(\alpha)$ implies that there exists ρ , $\sigma < \rho < \alpha$. But this is a contradiction, since then, $0(\sigma) = \tau < 0(\rho)$, contrary to the definition of τ .

Case (ii). As before let $\tau = \text{lub}\{0(\delta) \mid \delta < \alpha\}$. Then $\tau \leq 0(\alpha)$.

Suppose $\tau < 0(\alpha)$. If $\tau \leq \alpha$, then $I(\tau)$, for $\tau \leq \alpha \Rightarrow 0(\tau) \leq \tau \Rightarrow 0(\tau) = \tau$. But the fact that $I(\tau)$ leads directly to a contradiction, since there must be a σ , such that $0(\sigma) = \tau$; then there must be a ρ , $\sigma < \rho < \alpha$, and $0(\rho) > \tau$, contradicting the definition of τ .

So assume $\alpha < \tau < 0(\alpha)$. If $I(\tau)$ we are done by the argument above. If not, τ is either a gap ordinal or lies inside the gap of a gap ordinal. Arguing exactly as in Case (i) we see that this is an impossibility. Hence $\tau = 0(\alpha)$, and the function 0 is continuous at α , which had to be proved.

Chapter II

GAP-MINIMAL SYSTEMS OF NOTATIONS

Section 1

1.1 Definition. A system of notations R is a well-ordering of integers $<_R$.

If a system of notations is of order type α , then it assigns a unique notation to every ordinal smaller than α .

Let R be a system of notations. We write ' $|a|_R = \alpha$ ' just in case $a \in \text{Field}(R)$ & a is the α th element of R . We associate H-sets with R as follows:

$$H_R(a) = \emptyset \quad \text{if } |a|_R = 0$$

$$H_R(a) = H_R(b)^{(\omega)} \quad \text{if } |b|_R + 1 = |a|_R$$

$$H_R(a) = \{J(x,y) \mid y <_R a \text{ \& } x \in H_R(y)\} \quad \text{if } \text{Lim}(|a|_R)$$

The above definition of H-sets in terms of the ω -jump rather than in terms of ordinary jump is justified by the fact that we are interested in arithmetical degrees of sets in the constructible hierarchy, rather than in Turing degrees of the same sets. Since for any set A , A is 1-1 reducible to $A^{(\omega)}$ uniformly via $\lambda x J(x,0)$, our H-sets have the usual property that $a <_R b \Rightarrow H_R(a) \leq_1 H_R(b)$, uniformly in a and b .

In this chapter we study the arithmetical degrees of the H-sets associated with systems of notations and prove the main result of this thesis, namely the existence of gap-minimal systems of notations of arbitrary length ($<\omega_1^L$). The first part of the chapter contains a number of results necessary for the proof of the main theorem, and some of them interesting in their own right. They are mostly concerned with the relationship between H-sets and uniform upper bounds and with the position of certain H-sets in the constructible hierarchy.

Let α be a gap ordinal and let R be a well-ordering of integers all whose initial segments are in $M_\alpha \cap P(\omega)$. We will show that the H-sets associated with integers in R which are notations for ordinals smaller than the length (order-type) of R are themselves in $M_\alpha \cap P(\omega)$. Clearly, $H_R(0_R) = \emptyset$ is in $M_\alpha \cap P(\omega)$. Now, if $H_R(a) \in M_\alpha \cap P(\omega)$, and b is the R -successor of a , $H_R(b) = H_R(a)^{(\omega)}$. But since α is a limit ordinal, there exists $\beta < \alpha$, such that $H_R(a) \in M_\beta \cap P(\omega)$. By Lemme #0.2.5, $H_R(a)^{(\omega)} \in M_{\beta+1} \cap P(\omega) \subseteq M_\alpha \cap P(\omega)$, so $H_R(b) \in M_\alpha \cap P(\omega)$ also. Therefore if for some integer a in the field of R , $H_R(a)$ is not in $M_\alpha \cap P(\omega)$ and a is the R -least element with this property, a has to be a notation for a limit ordinal. We will show that such an a cannot exist. The result will be an easy corollary of the following:

1.2 Theorem. For all a, b in $\text{Field}(R)$, if a is in the field of $R_{b+R}1$, then $H_R(a)$ is Δ_1^1 in $R_{b+R}1$, uniformly in a .

In the statement of the theorem ' R_c ' stands for the initial segment of R determined by c , and ' $a +_R 1$ ' is an abbreviation for 'the unique element of R which is the R -immediate successor of a .'

1.3 Lemma.

- (a) If $A \in \Delta_1^1$, then $A' \in \Delta_1^1$, uniformly, i.e. there exists a recursive function f such that if e is a Δ_1^1 -index of A , then $f(e)$ is a Δ_1^1 -index for A' .
- (b) Let f be a recursive function such that for all x , $f(x)$ is a Δ_1^1 -index for a Δ_1^1 -set. Let A_x be the Δ_1^1 -set whose index is $f(x)$. Then the set $\{J(x,y) \mid x \in A_y\}$ is a Δ_1^1 -set, uniformly.
- (c) If A is in Δ_1^1 , then $A^{(\omega)}$ is in Δ_1^1 , uniformly.

Proof. Part (a) is a particular case of Theorem X, Chapter 16 in Rogers; part (b) appears in the same book as exercise 16-94 and it is easily proved. For (c), let e be a Δ_1^1 -index for A and let f be the recursive function found in (a). Let $g(n) = f^n(e)$, where $f^0(e) = e$, and $g(n+1) = f(g(n))$. g is clearly a recursive function and $g(n)$ is, by repeated applications of part (a), a Δ_1^1 -index for $A^{(n)}$. But $A^{(\omega)} = \{J(x,y) \mid x \in A^{(y)}\}$. Applying (b), we get the desired result.

The results of Lemma 1.3 hold in relativized form also. We shall not give the proof here, but we will use them freely in that form.

1.4 Lemma (Recursion Lemma, Rogers).

Let $<_R$ be a well-ordering of integers and let P be a binary relation on integers such that there exists a k , partial recursive function of two variables, such that for all integers e and for all a in the field of R ,

$$(b)(b <_R a \Rightarrow .P(b, \phi_e(b) \Rightarrow P(a, \phi_{k(e,a)}(a))).$$

Then there exists an r , such that for all a in the field of R , $P(a, \phi_r(a))$.

The proof of this lemma uses the recursion theorem and can be found in Rogers [16, Chapter XVI, p.398].

The proof of the theorem uses the recursion lemma and is a trivial modification of Kleene's result about the H -sets associated with 0 .

Proof of the Theorem.

Fix b in the field of R . Recall that we have to show that:

$$(a)(a \in \text{Field}(R_{b+1}) \Rightarrow H_{R_{b+1}}(a) \text{ is } \Delta_1^1 \text{ in } R_{b+1}, \\ \text{uniformly in } a).$$

We use the recursion lemma over the well-ordering R_{b+R}^1 , where P is taken to be

$$\{J(a,w) \mid a \in \text{Field}(R_{b+R}^1) \text{ \& } w \text{ is a } \Delta_1^{1,R_{b+R}^1} \text{ index for } H_R(a)\}.$$

Given a z such that $(c)(c <_{R_{b+R}^1} a \Rightarrow P(c, \phi_z(c)))$ we must show how to compute a w , uniformly in a and z such that $P(a,w)$.

Case 1. a is R -least. Let w be a Δ_1^{1,R_{b+R}^1} -index for \emptyset .

Case 2. a is an R -successor. Let c be the R -immediate predecessor of a . Then $\phi_z(c)$ is a Δ_1^{1,R_{b+R}^1} -index for $H_R(c)$. Since $H_R(a) = H_R(c)^{(\omega)}$, we can apply the construction in Lemma 1.3, to get a Δ_1^{1,R_{b+R}^1} -index for $H_R(a)$.

Case 3. a is an R -limit. Then

$$\begin{aligned} H_R(a) &= \{J(x,y) \mid y <_{R_{b+R}^1} a \text{ \& } x \in H_R(y)\} = \\ &= \{J(x,y) \mid y <_{R_{b+R}^1} a \text{ \& } (E f)(u) \sim T'_{1,1}(K\phi_z(y), f, x, u)\} \\ &= \{J(x,y) \mid y <_{R_{b+R}^1} a \text{ \& } (f)(Eu) T'_{1,1}(L\phi_z(y), f, x, u)\} \end{aligned}$$

where $T'_{1,1}$ is the Kleene T -predicate as defined in Rogers, 15.2.

For lack of space, we omitted the superscript R_{b+R}^1 from the T -predicates, but it belongs there.

But $\{y \mid y <_{R_{b+R^1}} a\}$ is arithmetical in R_{b+R^1} , uniformly in a . A Tarski-Kuratowski computation gives then $H_R(a)$ is in $\Delta_1^{1,R_{b+R^1}}$, and an index for it can be obtained uniformly in a .

1.5 Corollary. Let α be a gap ordinal and let R be a well-ordering of integers all whose initial segments are in $M_\alpha \cap P(\omega)$. Then for all a in the field of R , if $|a|_R < \alpha$, then $H_R(a) \in M_\alpha \cap P(\omega)$.

Proof. Let a be the R -least element of R such that $H_R(a)$ is not in $M_\alpha \cap P(\omega)$. By a remark above a has to be an R -limit. By the theorem $H_R(a)$ is Δ_1^1 in R_{a+R^1} , hence $H_R(a) \leq_T 0^{R_{a+R^1}}$. By hypothesis, R_{a+R^1} is in $M_\alpha \cap P(\omega)$. Since α is HYP, $0^{R_{a+R^1}} \in M_\alpha \cap P(\omega)$ also. Therefore $H_R(a)$ is in $M_\alpha \cap P(\omega)$. So there is no such a .

Remark that in the proof we used only the following property of the ordinal α : HYP(α). Therefore the conclusions of the theorem (and of the corollary) hold for any α having this property and such that $M_\alpha \cap P(\omega)$ contains a well-ordering of integers of the appropriate type.

Section 2.

In this section we investigate the connections between uniform upper bounds and H-sets. We will show that if α is a gap ordinal, then $H'(\alpha)$ is a uniform upper bound on $M_\alpha \cap P(\omega)$.

2.1 Lemma (Leeds). Let A be a uub for $M_\alpha \cap P(\omega)$, α a gap ordinal.

Then there exists a well-ordering of integers R_A such that:

- (i) Every initial segment of R_A is in $M_\alpha \cap P(\omega)$, and
- (ii) $R_A \leq_a A$.

Proof. Let W_e^A be the set of A -Gödel numbers of sets of integers in M_α . Since α is HYP, there exists a predicate $W(x)$ arithmetical in A such that

$$x \in W_e^A \text{ \& } W(x) \iff x \in W_e^A \text{ \& } W_x^A \text{ is a well-ordering.}$$

Let $P_A(a, b, x, y)$ be a predicate which holds just in case the following hold:

- (i) $a \in W_e^A \text{ \& } b \in W_e^A \text{ \& } W(a) \text{ \& } W(b)$
- (ii) $a < b \text{ \& } x \in \text{Field}(W_a^A) \text{ \& } y \in \text{Field}(W_b^A) \text{ \& } \vee a=b \text{ \& } J(x, y) \in W_a^A$.

Let

$$R_A = \{J(Jx, a), J(y, b) \mid P_A(a, b, x, y)\}$$

R_A is obviously arithmetical in A , and since it is constructed by joining together ω well-orderings, every initial segment in R is in M_α . The length of R_A is exactly α .

By the corollary to the previous theorem we can conclude that if b is in the field of R_A , then $H_{R_A}(b)$ is in $M_\alpha \cap P(\omega)$. But then it follows that $H_{R_A}(b) \leq_a A$ also.

Reviewing the situation up to now, we see that we can associate with every uniform upper bound on $M_\alpha \cap P(\omega)$ a well-ordering of integers which is arithmetical in the considered uub and moreover, such that the H-sets associated with that well-ordering are all in $M_\alpha \cap P(\omega)$. This is the case because we are inductively building sets along well-orderings in M_α , using sets already in M_α , and the closure properties of $M_\alpha \cap P(\omega)$ guarantee that the resulting sets will be in M_α too. Suppose now that we extend one of these well-orderings R_A in order to have a notation for α . One can ask then, where does $H(a)$ lie, if a is the notation we assigned to α . We will show that $H(a)$ cannot possibly be in M_α , and that $H(a)$ itself is a uub for $M_\alpha \cap P(\omega)$.

2.2 Proposition. Let A be a uub for $M_\alpha \cap P(\omega)$, α gap ordinal. Then there exists a well ordering of integers R'_A such that:

- (i) $R'_A \leq_a A$.
- (ii) Every initial segment of R'_A of length strictly smaller than α is in $M_\alpha \cap P(\omega)$.
- (iii) R'_A is of length $\alpha+1$.

Proof. Let d be an A -Gödel number for A . Since no uniform upper bound on $M_\alpha \cap P(\omega)$ can itself be in $M_\alpha \cap P(\omega)$, d is not an element of W_e^A . Therefore $P_A(d, d, x, y)$ will not hold for any choice of x and y . We add just one new element at the 'end' of R_A , therefore we must make sure that all the elements in R_A are smaller in the new ordering than the new element. Define R'_A as follows:

$$R'_A = R_A \cup \{J(x, J(0, Ld)) \mid x \in R_A\}$$

Since R_A is arithmetical in A , so is R'_A , R'_A is obviously of length $\alpha+1$, and every initial segment of R'_A of length strictly smaller than α is an initial segment of R_A of that length and by the construction of R_A is in $M_\alpha \cap P(\omega)$. Therefore R'_A is the desired well-ordering.

2.3 Proposition. Let R'_A be the well-ordering obtained above and let a be the last element of R'_A , i.e. the notation for α . Then $H_{R'_A}(a)$ is not an element of $M_\alpha \cap P(\omega)$.

Proof. Suppose $H_{R'_A}(a)$ is in $M_\alpha \cap P(\omega)$. We will show that we can construct a well-ordering of integers of length, which is arithmetical in $H_{R'_A}(a)$. But we know that for any two sets A and B , if A is of order β , and B is arithmetical in A , then B itself is of order at most β , so the constructed well-ordering will be in $M_\alpha \cap P(\omega)$ as well, contradicting the fact that $M_\alpha \cap P(\omega)$ contains no well-orderings of length α . By definition,

$$H_{R'_A}(a) = \{J(x,y) \mid y <_{R'_A} a \text{ \& } x \in H_{R'_A}(y)\}.$$

Define a well-ordering of integers S as follows:

$$S = \{J(x,y) \mid x \in H_{R'_A}(a) \text{ \& } y \in H_{R'_A}(a) \text{ \& } (Lx <_{R'_A} Ly <_{R'_A} a) \vee \\ \vee (Lx = Ly \text{ \& } Kx < Ky)\}$$

S is clearly a well-ordering of length greater than δ , for any $\delta < \alpha$, so S is of length at least α . In order to show that S is arithmetical in $H_{R'_A}(a)$ it is enough to observe that the following problems are uniformly arithmetical in $H_{R'_A}(a)$:

1. Determining if x is in the field of R'_A and $x <_{R'_A} a$:

$$x \in \text{Field}(R'_A) \text{ \& } x <_{R'_A} a \iff (Ey)(J(y,x) \in H_{R'_A}(a)).$$

2. Determining if $x <_{R'_A} y <_{R'_A} a$:

$$x <_{R'_A} y <_{R'_A} a \iff x \in \text{Field}(R'_A) \text{ \& } x <_{R'_A} a \text{ \& } y \in \text{Field}(R'_A) \text{ \& } \\ \text{\& } y <_{R'_A} a \text{ \& } y \neq x \text{ \& } \\ \text{\& } \{r \mid J(r,x) \in H_{R'_A}(a)\} \leq_a \{r \mid J(r,y) \in H_{R'_A}(a)\}.$$

Therefore S is arithmetical in $H_{R'_A}(a)$ and by the argument above it is in $M_\alpha \cap P(\omega)$. But this is a contradiction. Therefore $H_{R'_A}(a)$ is not a set of integers in M_α , if a is a notation in R'_A for α .

2.4 Theorem. Let α be a gap ordinal. There exists an arithmetical copy of $M_\alpha \cap P(\omega)$, E , such that $E \leq_a H_R(a)$, provided a is a notation in R for α .

Proof. The proof will be by induction on R , and we will in fact prove the stronger result that $H_R(a)_{T_a} \geq E_a$.

Recall that we have available the operation $*$ which enables us to build an arithmetical copy of $M_{\alpha+1}$ given an arithmetical copy of M_α , and that the new arithmetical copy is uniformly recursive in the ω th jump of the old arithmetical copy (that of M_α). Let E be a recursive arithmetical copy of M_ω . It is useful to pick E to be O_R -initial. For example pick E to be:

$$\{J(J(m, O_R), J(n, O_R)) \mid (Ex)(Ey)(G(x) = m \ \& \ G(y) = n \ \& \ x \in y)\}$$

where G is the recursive function which maps the hereditarily finite sets one-to-one onto the natural numbers. Now define the desired copy E_a of M_α by induction over R as follows:

$$E_{O_R} = E$$

$$E_b = (E_c, b)^* \quad \text{if } |b|_R = |c|_R + 1$$

$$E_b = \bigcup_{c <_R b} E_c \quad \text{if } |b|_R \text{ is a limit ordinal}$$

$$E_a = \bigcup_{b <_R a} E_b$$

E_a is clearly an arithmetical copy of M . We show now by induction that E_b is recursive in $H_R(b)$, uniformly in b . E_{0_R} is recursive, hence recursive in $\emptyset = H_R(0_R)$. Suppose now that $b <_R a$ and that $E_b \leq_T H_R(b)$. Let c be the R -successor of b . Then,

$$E_c = (E_b, c)^* \leq_T E_b^{(\omega)} \leq_1 H_R(b)^{(\omega)} = H_R(c)$$

Since both reducibilities above are uniform, it follows that E_c is uniformly recursive in $H_R(c)$. In order to prove the result at limits we have to examine the operation $*$ in greater detail. Recall that if $E_c = (E_b, c)^*$, then we have:

$$E_c = E_b \cup \{J(d, J(y, c)) \mid y \text{ is the least Gödel number of some formula } \phi \text{ and } \phi \text{ defines a new set and } d \in \text{Field}(E_c) \text{ and } E_c \models \phi d\}$$

Hence if we define the E 's along some well-ordering of integers, the way we did above, they will form an increasing sequence of sets.

Observe that given an x in some E we can tell the level at which the set represented by x first appeared in the constructible hierarchy, for it is enough to compute LLx ; the set represented by x first has an image in E_{LLx} , and therefore it is new in $M_{\omega+|LLx|_R}$.

Let b be an R -limit. We will show that E_b is recursive in the (ordinary) jump of $H_R(b)$. By assumption $E_c \leq_T H_R(c)$, for all $c <_R b$. Recall that we have used in a previous proof the fact that if b is an R -limit, the predicate ' x is in the field of R and

$x <_R b'$ is uniformly decidable in $H_R(b)'$. Rather than give an explicit expression for the function, we indicate how we can decide membership in E_b given an oracle for $H_R(b)'$. Given a number x , we want to know whether x is in E_b . Compute LLx . Ask if $LLx \in \text{Field}(R)$ and $LLx <_R b$. If the answer is 'no,' x is not in E_b . If the answer is 'yes,' ask if x is in E_{LLx} . Since by the induction hypothesis $E_{LLx} \leq_T H_R(LLx)$, and $H(LLx) \leq_1 H_R(b) \leq_T H_R(b)'$, given an oracle for $H_R(b)'$ we can decide membership in E_{LLx} .

Remark that although at limit ordinals we have arithmetical rather than Turing reducibility, the induction still works, because finite jumps are absorbed by ω -jumps. For example, let b be an R -limit and let c be the R -successor of b ; suppose $E_b \leq_T H_R(b)'$. Then

$$E_c = (E, c)^* \leq_T E_b^{(\omega)} \leq_1 (H(b)')^{(\omega)} \equiv_T H_R(b)^{(\omega)} = H_R(c)$$

So the induction can continue from this point on as before. We have therefore shown that there exists an arithmetical copy of M_α which is recursive in $H_R(a)'$.

From this the following easy, but important corollary is immediate:

2.5 Corollary. $H_R(a)'$ is a uniform upper bound for $M_\alpha \cap P(\omega)$.

Proof. Since E_a is an arithmetical copy of M_α , E_a is a uniform upper bound for $M_\alpha \cap P(\omega)$. But clearly any set in which a uniform upper bound for a collection of sets of integers is recursive in, is itself a uniform upper bound for that collection, so we are done.

It can be shown (using the same kind of proof as in the above theorem) that if R is chosen to be R_A , and A is a uniform upper bound for $M_\alpha \cap P(\omega)$, then E_a itself is arithmetical in A .

Section 3

In this section we build the gap-minimal systems of notations extending arbitrarily close to ω_1^L .

3.1 Definition. Let A be a uniform upper bound for a countable collection of sets of integers \mathcal{C} . A is said to be a β -least uniform upper bound for \mathcal{C} if and only if A is arithmetical in the β -th jump of any other uniform upper bound for \mathcal{C} .

In other words, if we denote the collection of uniform upper bounds on \mathcal{C} by $U(\mathcal{C})$,

$$A \in U(\mathcal{C}) \Rightarrow A \text{ is } \beta\text{-least} \iff (B)(B \in U(\mathcal{C}) \Rightarrow A \leq_a B^{\omega \cdot \beta}).$$

Our aim is to define a sequence of H-sets which is arithmetically minimal in the following sense: If R is any system of notations, then $H^{\min}(a) \leq_a H_R(a')$, provided a and a' are notations for the same ordinal. The sequence H^{\min} is defined using our previous result connecting H-sets with uniform upper bounds and a series of results in Putnam & Leeds [10], which we quote:

Theorem A. Let α be a gap ordinal and let the length of the gap at α be $g(\alpha)$. Any complete set $E_{\alpha+g(\alpha)}$ of order $\alpha+g(\alpha)$ is a $g(\alpha)$ -least uniform upper bound on $M_\alpha \cap P(\omega) = M_{\alpha+g(\alpha)} \cap P(\omega)$.

Theorem B. Let α be a gap ordinal and let the length of the gap at α be $g(\alpha)$. There is no γ -least uniform upper bound on $M_\alpha \cap P(\omega)$, if $\gamma < g(\alpha)$.

Theorem C. Let α be a gap ordinal and let the length of the gap at α be $g(\alpha)$. Then there exists a $g(\alpha)$ -least uniform upper bound on $M_\alpha \cap P(\omega)$, A , such that both A and a $g(\alpha)^{\text{th}}$ jump of A are in $M_{\alpha+g(\alpha)}$.

The proof of Theorem A is not very hard and is similar to our proof of Theorem 2.4. The main idea of the proof is that given any uniform upper bound for $M_\alpha \cap P(\omega)$, we can build an arithmetical copy of M_α which is arithmetical in the $g(\alpha)^{\text{th}}$ jump of the uniform

upper bound. The proof of Theorem B is much more delicate and it involves a forcing argument. The 'ancestor' of this proof is a theorem by Boyd which states that if \mathcal{A} is a countable ω -model for analysis, then there exist two sets of integers A and B such that A and B are uniform upper bounds on \mathcal{A} , and such that if C is any set of integers which is arithmetical both in A and in B , then C is second-order definable over the sets of integers in \mathcal{A} . It followed from that theorem that there was no arithmetically least uniform upper bound on \mathcal{A} ; for if K were such a uniform upper bound, then $K \leq_a A$, $K \leq_a B$, and K would be second-order definable over \mathcal{A} , so $K \in \mathcal{A}$, and K' (which is in \mathcal{A}) would be recursive in K , which is a contradiction. Theorem C is a corollary of Theorem B and of the definability of forcing over our ground model. We stress again the fact that our results will depend very strongly on Theorem B, and a real understanding of them cannot be achieved without a detailed examination of that proof.

We list now the necessary (previously proved) facts for the definition of the minimal sequence of H-sets. From now on α will be a fixed arbitrary gap ordinal and the length of the gap at α will be denoted by $g(\alpha)$. Recall that $g(\alpha) \geq 1$. Let $U(\alpha)$ denote the collection of uniform upper bounds on $M_\alpha \cap P(\omega)$. Clearly $U(\alpha) \cap M_\alpha \cap P(\omega) = \emptyset$. Then the following are true:

(i) If A is in $U(\alpha)$ then there exists a well-ordering R_A of length $\alpha+1$ such that:

- a. All initial segments of R_A of length strictly smaller than α are in $M_\alpha \cap P(\omega)$.
- b. $R_A \leq_a A$ and $H_{R_A}(a) \leq_a A$, if a is the α th element of R_A .

(ii) If K is a complete set of order $\alpha + g(\alpha)$, then $K \in U(\alpha)$ and K is $g(\alpha)$ -least.

(iii) If a is a notation for α in some system of notations R , then $H_R(a)'$ is a uniform upper bound for $M_\alpha \cap P(\omega)$.

(iv) If A and B are two sets of integers such that for some n , $A = B^{(n)}$, then $A^{(\omega)} \equiv B^{(\omega)}$.

The main theorem of this chapter will assert the existence of a system of notations which is 'best possible' with respect to the arithmetical degrees associated with its H -sets. We now make precise the notion of 'best possible.' Consider now an arbitrary system of notations (i.e. a well-founded partial-ordering of integers), as defined by Enderton. Such a system, R , is said to be minimal if for any other system of notations, S , if $a \in \text{Field}(R)$ and $b \in \text{Field}(S)$ and $|a|_R = |b|_S$, then $H_R(a) \leq_T H_S(b)$. For example, O is a system of notations which is minimal and contains notations for all constructive ordinals. Putnam and Luckham [11] have shown that there is no minimal system of notations containing a notation for constructive ω_1 . In

[12], Putnam and Lukas relax the requirement on minimality a little bit further and define a system of notations R to be almost-minimal if for every system of notations S , if $a \in \text{Field}(R)$ and $b \in \text{Field}(S)$ and $|a|_R = |b|_S$ then $H_R(a)$ is arithmetical in $H_S(b)$. In the above definitions, the H -sets are the usual H -sets, defined using the ordinary jump. The main result of the Putnam and Lukas paper cited above is that there are almost-minimal systems of notations containing notations for all ordinals $< \beta_0$, and that in fact there is a system of notations of length β_0 . We return now to systems of notations which are well-orderings of integers and in which the H -set associated with a notation for a successor ordinal is the ω -jump of the H -set associated with the predecessor of that ordinal. Let R be such a system of notations. We will say that R is gap-minimal if for any system S of the same type, if $a \in \text{Field}(R)$ and $b \in \text{Field}(S)$ and $|a|_R = |b|_S$ and $\text{Lim}(|a|_R)$, then:

$$H_R(a) \leq_a H_S(b)^{h(\alpha)}$$

where h is a function defined as follows:

$$h(\gamma) = (\text{least index greater than or equal to } \gamma) - \gamma$$

and α is the ordinal whose notation in R is a and whose notation in S is b .

It follows from the definition of gap-minimality that the arithmetical degrees assigned to H -sets along a gap-minimal system of notations are as low as possible, i.e. for no $\beta < h(\gamma)$, can $H_R(a)$

be β -least, because that would contradict the non-existence of β -least uniform upper-bounded asserted by Theorem B.

3.2 Main Theorem

Let $\alpha < \omega_1^L$. There exists a gap-minimal system of notations containing notations for all ordinals less than α . Clearly it is enough to consider only limit ordinals in the proof of the theorem and a moment of thought will show that it is enough to consider only those limit ordinals which are gaps. The proof of the theorem will be by induction.

Induction hypothesis: For every gap ordinal less than α , β , there exists a gap-minimal system of notations of length β . We will consider three cases, which obviously include all gap ordinals:

Case I. There are no gap ordinals smaller than α ($\alpha = \beta_0$).

Case II. There is a greatest gap ordinal smaller than α .

Case III. There is no greatest gap ordinal less than α , i.e. α is a limit of gap ordinals, and $\alpha \neq \beta_0$.

The proofs of the induction step in Cases I, II and III will be contained in Propositions I, II and III, respectively.

Proposition I. There exists a gap-minimal system of notations of length β_0 .

Proof. We will use Theorem 5 in Putnam and Lukas which states that for every ordinal less than or equal to β_0 , β there exists an almost-minimal system of notations of length $\omega.\beta$. In our proof we will denote H-sets obtained along a well-ordering using ordinary jumps by capital Roman letters (H) and H-sets obtained along the same well-ordering using ω -jumps by capital script letters (\mathcal{H}). Let R be a branch of length $\omega.\beta_0$ of the system of notations whose existence is asserted by the Putnam-Lukas theorem. R is then an almost-minimal system of notations. Let K be the generic uniform upper bound on $M_{\beta_0} \cap P(\omega)$. Let S be the well-ordering which is arithmetical in K and such that $H_S(\beta_0)$ is arithmetical in K . We know that $H_S(\beta_0)$ is 1-least. Claim that R_{β_0} is a gap minimal system of notations. In this proof we write $H_R(\alpha)$, to mean $H_R(a)$, where $|a|_R = \alpha$. For all $\alpha < \beta_0$, $h(\alpha) = 0$. We want to show that for any system of notations T , for all $\alpha < \beta_0$, $\mathcal{H}_R(\alpha) \leq_a \mathcal{H}_T(\alpha)$. By almost minimality $H_R(\alpha) \leq_a H_T(\alpha)$, for all $\alpha < \omega.\beta_0$. In particular $H_R(\omega.\alpha) \leq_a H_T(\omega.\alpha)$. (Use $\alpha < \beta \iff \omega.\alpha < \omega.\beta$) It is easy to see that $\mathcal{H}_R(\alpha) \equiv_a H_R(\omega.\alpha)$. By transitivity of \leq_a , it follows that $\mathcal{H}_R(\alpha) \leq_a \mathcal{H}_T(\alpha)$, for all $\alpha < \beta_0$. In particular, $\mathcal{H}_R(\alpha) \leq_a \mathcal{H}_S(\alpha)$. Therefore $\mathcal{H}_R(\beta_0) \leq_a \mathcal{H}_S(\beta_0) \leq_a K \leq_a K^{(\omega)}$. So $\mathcal{H}_R(\beta_0)$ is a 1-least uniform upper bound on $M_{\beta_0} \cap P(\omega)$, i.e. R behaves as desired at its

β_0 th point and below. Therefore the initial segment of R , which is of length β_0 is a gap-minimal system of notations.

Proposition II.

Suppose α is a gap ordinal and suppose there is a largest gap ordinal smaller than α . Assume that there is a gap-minimal system of notations R giving a notation to all ordinals smaller than or equal to $\gamma + g(\gamma)$, where γ is the greatest gap ordinal less than α . We will show how to obtain a gap-minimal system of notations extending up to $\alpha + g(\alpha)$. Let K be the uniform upper bound on $M_\alpha \cap P(\omega)$ obtained using Theorem C. Then there is a $g(\alpha)$ -jump of K which is arithmetical in the complete set of order $\alpha + g(\alpha)$. Observe that it is always possible to "paste together" two well-orderings in order to obtain a new well-ordering whose order-type is the sum of the order-types of the two components, provided the components are disjoint. Our proof will proceed as follows:

(i) Given the "generic" uniform upper bound K , build the "generic" well-ordering W , of length $g(\alpha)$ along which the arithmetically low $K^{g(\alpha)}$ is obtained.

(ii) Build the well-ordering R_K as indicated in Theorem ; R_K is then arithmetical in K , $H_{R_K}(a) \leq_a K$ for all $a \in \text{Field}(R_K)$, $|a|_{R_K} < \alpha$, and $H_{R_K}(a)'$ is a $g(\alpha)$ -least uniform upper bound on $M_\alpha \cap P(\omega)$.

(iii) Obtain disjoint copies of R_K and W and paste them together to obtain a well-ordering S of length $\alpha + g(\alpha)$ as follows:

$$\begin{aligned} S = & \{J(J(x,0),J(y,0)) \mid J(x,y) \in R_K\} \cup \\ & \{J(J(x,1),J(y,1)) \mid J(x,y) \in W\} \cup \\ & \{J(J(x,0),J(y,1)) \mid x \in \text{Field}(R_K) \text{ \& } y \in \text{Field}(W)\} \end{aligned}$$

(iv) Obtain disjoint copies of R and S . Delete the initial segment of length $\gamma + g(\gamma)$ of S and replace it by R . Let

$|a|_S = \gamma + g(\gamma)$. Construct the well-ordering T of length $\alpha + g(\alpha)$:

$$\begin{aligned} T = & \{J(J(x,0),J(y,0)) \mid J(x,y) \in R\} \cup \\ & \{J(J(x,1),J(y,1)) \mid J(x,y) \in S \text{ \& } (J(a,x) \in S \vee x = a)\} \cup \\ & \{J(J(x,0),J(y,1)) \mid x \in \text{Field}(R) \text{ \& } y \in \text{Field}(S) \text{ \& } \\ & \quad \& (J(a,y) \in S \vee y = a)\} \end{aligned}$$

We claim that T is a gap-minimal system of notations of length $\alpha + g(\alpha)$. First observe that by taking a copy of a well-ordering in the manner above, we do not alter the arithmetical degrees of the H -sets constructed along the well-ordering. Then, by the induction hypothesis,

$$T_{J(a,1)} = \{J(x,y) \in T \mid J(y,H(a,1)) \in T\},$$

the initial segment of T of length $\gamma + g(\gamma)$ is a gap-minimal system of notations. Observe that $H_T(b)'$ is a $g(\gamma)$ -least uniform upper bound on $M_Y \cap P(\omega)$ (where $b = J(a,1)$). For if we construct the generic well-ordering for $M_Y \cap P(\omega)$, W_1 , $H_{W_1}(c)$ will be a $g(\gamma)$ -least uniform upper bound for $M_Y \cap P(\omega)$, provided c is a notation in

W_1 for γ . But by gap-minimality $H_T(b) \leq_a H_{W_1}(c)^{g(\gamma)} \leq_a E_{\gamma+g(\gamma)}$

(the complete set of order $\gamma+g(\gamma)$). Therefore if V is any other well-ordering and v is a notation in V for γ ,

$H_T(b) \leq_a E_{\gamma+g(\gamma)} \leq_a H_V(v)^{g(\gamma)}$. By gap-minimality again,

$H_T(J(a,1)) \leq_a H_S(a)^{h(\gamma+g(\gamma))}$. Since $\gamma+g(\gamma)$ is an index, $h(\gamma+g(\gamma)) = 0$,

so we have $H_T(J(a,1)) \leq_a H_S(a)$. Since the initial R_K was generic, the

H -set built along it at level α is an $g(\alpha)$ -least uniform upper bound

on $M_\alpha \cap P(\omega)$ as observed in (ii). T is exactly the same as R_K on

its terminal stretch, so we know that the gap-minimality property holds

at the new gap, α . The problem remains then to show that T is gap

minimal on the interval $(\gamma+g(\gamma), \alpha)$. In order to do this we prove an

easy lemma:

Lemma. Let W be a well-ordering of integers. For an arbitrary set of integers A and $a \in \text{Field}(W)$, define a set A_W^a as follows:

$$A_W^a = A \quad \text{if} \quad |a|_W = 0$$

$$A_W^a = A_W^{b(\omega)} \quad \text{if} \quad |b|_W + 1 = |a|_W$$

$$A_W^a = \{J(x,y) \mid y <_W a \text{ \& } x \in A_W^y\} \quad \text{if} \quad \text{Lim}(|a|_W)$$

Let A and B be two sets of integers such that $A \leq_a B$. Then, if

$a \in \text{Field}(W)$ and $|a|_W \neq \emptyset$, $A_W^a \leq_1 B_W^a$, uniformly in a . The Lemma

really says that if we start with one set arithmetical in another and

take jumps of both along the same well-ordering, at the same point,

the jump of the first set will always be of lower $l-1$, hence

hence arithmetical also, degree, than the jump of the second set. We sketch the proof for the notations in W up to and including the first limit ordinal, ω . The rest of the proof is similar.

$$A \leq_a B \iff (\exists n)(A \leq_1 B^{(n)}).$$

Let $f(x)$ be the recursive function which takes $A^{(\omega)}$ 1-1 into $B^{(\omega)}$.

Of course the f we pick will depend upon the finite jump of B we pick at the beginning as being above A . If a is a notation in W for a finite ordinal, say m , then A^a is carried 1-1 into B^a by f iterated m times. Suppose now that a is a notation in W for α . Then

$$\begin{aligned} J(x, y) \in A^a &\iff y <_W a \text{ \& } x \in A^y \iff y <_W a \text{ \& } f|y|_W(x) \in B^y \\ &\iff J(f|y|_W(x), y) \in B^a \end{aligned}$$

So $A^a \leq_1 B^a$.

Since we already know that for any well-ordering W , $H_T(\gamma + g(\gamma)) \leq_a H_W(\gamma + g(\gamma))$, and that $h(\beta) = 0$ for any ordinal between the end of the gap at γ and α , it follows by the Lemma that T is well-behaved in that interval. Moreover $H_T(\alpha)$ is arithmetically equivalent to $H_{R_K}(\alpha)$, therefore it is a $g(\alpha)$ -least uniform upper bound of $M_\alpha \cap P(\omega)$. Therefore T is a gap-minimal system of notations which gives notations to all ordinals smaller than $\alpha + g(\alpha)$. By Theorem B it also follows that T assigns arithmetical degrees to its H -sets which are as low as possible.

In the above proof we wrote $H_R(\alpha)$, for $H_R(a)$, if $|a|_R = \alpha$. No possible confusion can arise, since our systems give unique notations to ordinals.

Proposition III

Suppose now that α is a gap ordinal which is a limit of gap ordinals, i.e. that there is no greatest gap ordinal less than α . In order to prove that there exists a gap minimal system of notations extending up to α , given the possibility of building such systems up to any smaller gap ordinal, we need to prove the following lemma:

Lemma. Let α be a gap ordinal which is a limit of gap ordinals and let K be the generic uniform upper bound on $M_\alpha \cap P(\omega)$ given by Theorem C. Let R be the generic well-ordering associated with $K^{g(\alpha)}$. (R is the well-ordering obtained by pasting together the well-ordering along which the "good" $g(\alpha)$ -jump of K is taken). Then we can find an ω -sequence of gap-ordinals converging to α , uniformly in $H_R(a)'$, where a is the notation of α in R .

Proof. The strategy is to define a function from the integers into the field of R which is an order-preserving function and whose values converge to a , and to do this effectively in $H_R(a)'$. We first show that we can pick out among the notations in R , those notations which correspond to gap ordinals. Recall that the following problems are uniformly arithmetical in $H_R(a)$;

1. Deciding whether $x \in \text{Field}(R)$ & $x <_R a$.

2. Deciding whether $x <_R y <_R a$.

In the proof of the theorem stating that $H_R(a)'$ is a uniform upper bound on $M_\alpha \cap P(\omega)$, we built a sequence of arithmetical copies of M_β 's, for $\beta < \alpha$, indexed by elements of R . It followed by the construction that for $x <_R a$, E_x was recursive in $H_R(x)'$ uniformly in x , which in turn is 1-1 reducible uniformly in x to $H_R(a)$. Therefore the following two problems are decidable uniformly in $H_R(a)'$ (arithmetically in $H_R(a)$).

3. Deciding if $x \in \text{Field}(R)$ and if x is a notation for an index ordinal less than α :

$$x <_R a \text{ \& } I(|x|_R) \iff (Ey)(y \text{ represents a set of integers and } y \notin E_x \text{ and } (Ez)(y \in E_z \text{ \& } y <_R z <_R a \text{ \& } \sim(Eu)(y <_R u <_R z))).$$

4. Deciding if $x \in \text{Field}(R)$ and if x is a notation for a gap ordinal.

We will write in words what the predicate says, since its complicated form obscures the very simple meaning:

$$x <_R a \text{ \& } G(|x|_R) \iff x \text{ is a notation in } R \text{ for a limit and } x \text{ is a limit of indices and } x \text{ is not a notation for an index.}$$

Having available these decision procedures we build now the desired ω -sequence of gaps converging to α . First we extract from R a well-ordering containing only notations for gaps:

$R' = \{J(x,y) \mid J(x,y) \in R \text{ and } x \text{ and } y \text{ are notations for gap ordinals}\}.$

By 4, $R' \leq_a H_R(a)$. Now we define the R' -valued function which we need:

Let g be defined as follows:

$$g(0) = \mu z[z <_{R'} a]$$

$$g(n+1) = \mu z[z \notin \{g(0), \dots, g(n)\} \text{ \& } z <_{R'} a]$$

g is clearly arithmetical in $H_R(a)$. (In fact it is recursive in $H_R(a)$.) We are trying to define an f with the properties

- (i) $(x)(f(x) <_{R'} a)$ and
- (ii) $(x)(y)(x < y \Rightarrow f(x) <_{R'} f(y))$.

Let f be:

$$f(0) = g(0)$$

$$f(n+1) = g(z), \text{ where } z = \mu z'[f(n) <_{R'} g(z')].$$

It is easy to see that f satisfies (i) and (ii), that f is arithmetical in $H_R(a)$ and that the sequence $f(n)$ converges in R' to a , as n gets bigger and bigger. Therefore the lemma is proved.

Let β_n be the ω -sequence of gaps converging to α . By the induction hypothesis we can find an ω -sequence of gap minimal systems of notations $\{R_n\}$ $n \in \mathbb{N}$ such that R_n gives a notation to every ordinal less than β_n . Using the R_n 's we will build a gap-minimal

system of notations extending to α . First we build disjoint copies of the R_n 's as follows:

$$R'_n = \{J(J(x,n), J(y,n)) \mid J(x,y) \in R_n\}$$

Clearly for all n , if $a \in \text{Field}(R_n)$, $H_{R_n}(a) \equiv_T H_{R'_n}(a)$. Define inductively a sequence of well-orderings T_n . Let $|b_n|_{R'_{n+1}} = \beta_n$.

$$T_0 = R'_0$$

$$\begin{aligned} T_{n+1} = & T_n \cup \{J(x,y) \in R'_{n+1} \mid J(b_n, x) \in R'_{n+1} \vee x = b_n\} \\ & \cup \{J(x,y) \mid x \in \text{Field}(T_n) \ \& \ y \in \text{Field}(R'_{n+1}) \\ & \ \& \ (y = b_n \vee J(b_n, y) \in R'_{n+1})\} \end{aligned}$$

Clearly for every n , $T_n \subseteq T_{n+1}$, and T_{n+1} is an extension (proper) of T_n which is compatible with T_n , i.e. no element which is added to the field of T_n to form the field of T_{n+1} is already in T_n . By an argument similar to that used in the successor case we see that for every n , T_n is a gap-minimal system of notations containing notations for every ordinal less than β_n .

Let $T = \bigcup_{n \in \mathbb{N}} T_n$. We claim that T is the desired gap-minimal system of notations. The length of T is greater than δ , for any $\delta < \alpha$. It follows that the length of T is exactly α . By construction T is arithmetical in $H_R(a)$, where R was the generic well-ordering given by Theorem C. To see that TT is gap-minimal it is enough to observe that for any $\beta < \alpha$, $H_{R'_n}(b) \equiv_a H_T(c)$, if

if $|b|_{R'_n} = |c|_T = \beta$. Let β be an arbitrary ordinal less than α .

Then for some n , $\beta_n \leq \beta < \beta_{n+1}$, since $\beta_n \uparrow \alpha$. Let S be any system of notations of length greater than or equal to β . Suppose

$|b|_T = |c|_{R'_{n+1}} = |d|_S = \beta$. By minimality of R'_{n+1} ,

$H_{R'_{n+1}}(c) \leq_a H_S(d)^{h(\beta)}$. But by the observation above, $H_{R'_{n+1}}(c) \equiv_a H_T(b)$.

Therefore T is well-behaved at β , which had to be proved.

The induction is thus completed and we have proved the Theorem.

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